Exploring Proportional Relationships:
The Case of Mr. Donnelly

Mr. Donnelly wanted his students to understand that quantities in a proportional (multiplicative) relationship grow at a constant rate and that three key strategies are useful for solving problems of this type—scaling up, scale factor, and unit rate. He selected the Candy Jar task for the lesson since it was aligned with his goals, was cognitively challenging, and had multiple entry points.

> A candy jar contains 5 Jolly Ranchers (JRs) and 13 jawbreakers (JBs). Suppose you had a new candy jar with the same ratio of Jolly Ranchers to jawbreakers, but it contained 100 Jolly Ranchers. How many jawbreakers would you have? Explain how you know.

As students began working with their partners on the task, Mr. Donnelly walked around the room, stopping at different groups to listen in on their conversations and to ask questions as needed—for example, “How did you get that?” “How do you know that the new ratio is equivalent to the initial ratio?” When students struggled to figure out what to do, he encouraged them to look at their work from the previous day, which included producing a table of ratios equivalent to 5 JRs:13 JBs and a unit rate of 1 JR to 2.6 JBs. He also encouraged students to consider how much bigger the new candy jar must be when compared to the original jar.

As he made his way around the room, Mr. Donnelly also made note of the strategies that students were using (see reverse side) so he could decide which groups he wanted to present their work. After visiting each group, he decided that he would ask groups 4, 5, and 2 to share their approaches (in this order) since each of these groups had used one of the strategies that he was targeting, and this sequencing would reflect the sophistication and frequency of strategies.

During the discussion, Mr. Donnelly asked the presenters (one student from each of the targeted groups) to explain what their groups had done and why, and he invited other students to consider whether the approach made sense and to ask questions. He made a point of labeling each of the three strategies, asking students which strategy was most efficient in solving this particular task, and posing questions to help students make connections among the different strategies and with the key ideas that he was targeting. Specifically, he wanted students to see that that the scale factor identified by group 5 was the same as the number of entries in the table created by group 4 (or the number of small candy jars that it would take to make the new candy jar) and that the unit rate identified by group 2 was the factor that connected the JRs and JBs in each row of the table.

Below is an excerpt from the discussion that grew out of the unit rate solution presented by Jerry from group 2.

Jerry: We figured that there was 1 JR for 2.6 JBs so that a jar with 100 JRs would have 260 JBs. So we got the same thing as the other groups.
Mr. D.: Can you tell us how you figured out that there was 1 JR for 2.6 JBs?
Jerry: We divided 13 by 5.
Mr. D.: Does anyone have any questions for Jerry? Danielle?
Danielle: How did you know to do 13 ÷ 5?
Jerry: See, we wanted to find out the number of JBs for 1 JR. So if we wanted JRs to be 1, we needed to divide the number of JRs by 5. So now we needed to do the same thing to the JBs.
Danielle: So how did you then get 260 JBs?
Jerry: Well, once we had 1 JR to 2.6 JBs, it was easy to see that we needed to multiply each type of candy by 100 so we could get 100 JRs.

Written by Margaret Smith (University of Pittsburgh), drawing on two sources: *Principles to Actions: Ensuring Mathematical Success for All* (Reston, Va.: National Council of Teachers of Mathematics, 2014) and *Improving Instruction in Rational Numbers and Proportionality: Using Cases to Transform Mathematics Teaching and Learning,* vol. 1, by Margaret S. Smith, Edward A. Silver, and Mary Kay Stein (New York: Teachers College Press, 2005). This activity is intended to support the Guiding Principle for Teaching and Learning in *Principles to Actions: Ensuring Mathematical Success for All* (NCTM 2014).

Mr. D.: So Jerry's group multiplied by 100, but Danielle and her group (group 5) multiplied by 20. Can they both be right? Amanda?

Amanda: Yes. Jerry's group multiplied 1 and 2.6 by 100, and Danielle and her group multiplied 5 and 13 by 20. Jerry's group multiplied by a number 5 times bigger than Danielle's group because their ratio was 1/5 the size of the ratio Danielle's group used. So it is the same thing.

Mr. D.: Do others agree with what Danielle is saying? [*Students nod their heads and give Danielle a thumbs-up.*] So what is important here is that both groups kept the ratio constant by multiplying both the JRs and JBs by the same amount. We call what Jerry and his group found the *unit rate*. A unit rate describes how many units of one quantity (in this case JBs) correspond to one unit of another quantity (in this case JRs).

[*Mr. Donnelly pauses to write this definition on the board.*]

Mr. D.: I am wondering if we can relate the unit rate to the table that group 4 shared. Take two minutes and talk to your partner about this. [*The students work for two minutes before Mr. Donnelly reopens the discussion by turning to two students from group 4.*]

Mr. D.: Kamiko and Jerilyn, can you tell us what you were talking about?

Kamiko: We noticed that if we looked at any row in our table that the number of JBs in the row was always 2.6 times the number of JRs in the same row.

Mike: Yeah, we saw that too. So it looks like any number of JRs times 2.6 will give you the number of JBs.

Mr. D.: So what if we were looking for the number of JBs in a jar that had 1000 JRs?

Mike: Well, the new jar would be 1000 times bigger, so you multiply by 1000.

Mr. D.: So take 2 minutes and see if you and your partner can write a rule that we could use to find the number of JBs in a candy jar no matter how many JRs are in it.

[*After two minutes the discussion continues.*]

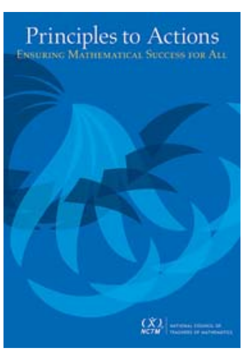
Toward the end of the lesson, Mr. Donnelly placed the solution produced by group 1 on the document camera and asked students to decide whether or not it was a viable approach to solving the task and to justify their answer. He told them that they would have five minutes to write a response that he would collect as they exited the room. He thought that this would give him some insight into whether or not individual students were coming to understand that ratios needed to grow at a constant rate that was multiplicative, not additive.

<table>
<tr><th>Group 1
(incorrect, additive)</th><th>Groups 3 and 5
(scale factor)</th><th>Groups 4 and 7
(scaling up)</th></tr>
<tr>
<td>100 JRs is 95 more than the 5 we started with. So we will need 95 more JBs than the 13 we started with.

5 JRs + 95 JRs = 100 JRs
13 JBs + 95 JBs = 108 JBs</td>
<td>You had to multiply the five JRs by 20 to get 100, so you'd also have to multiply the 13 JBs by 20 to get 260.

(× 20)
5 JRs → 100 JRs
13 JBs → 260 JBs
(× 20)</td>
<td><table>
<tr><th>JR</th><th>JB</th><th>JR</th><th>JB</th></tr>
<tr><td>5</td><td>13</td><td>55</td><td>143</td></tr>
<tr><td>10</td><td>26</td><td>60</td><td>156</td></tr>
<tr><td>15</td><td>39</td><td>65</td><td>169</td></tr>
<tr><td>20</td><td>52</td><td>70</td><td>182</td></tr>
<tr><td>25</td><td>65</td><td>75</td><td>195</td></tr>
<tr><td>30</td><td>78</td><td>80</td><td>208</td></tr>
<tr><td>35</td><td>91</td><td>85</td><td>221</td></tr>
<tr><td>40</td><td>104</td><td>90</td><td>234</td></tr>
<tr><td>45</td><td>117</td><td>95</td><td>247</td></tr>
<tr><td>50</td><td>130</td><td>100</td><td>260</td></tr>
</table></td>
</tr>
<tr><th>Group 2 (unit rate)</th><th colspan="2">Group 6 (scaling up)</th></tr>
<tr>
<td>Since the ratio is 5 JRs for 13 JBs, we divided 13 by 5 and got 2.6. So that would mean that for every 1 JR there are 2.6 JBs. So then you just multiply 2.6 by 100.
(× 100)
1 JR → 100 JRs
2.6 JBs → 260 JBs
(× 100)</td>
<td colspan="2"><table>
<tr><td>JRs</td><td>5</td><td>10</td><td>20</td><td>40</td><td>80</td><td>100</td></tr>
<tr><td>JBs</td><td>13</td><td>26</td><td>52</td><td>104</td><td>208</td><td>260</td></tr>
</table>
We started by doubling both the number of JRs and JBs. But then when we got to 80 JRs, we didn't want to double it anymore because we wanted to end up at 100 JRs, and doubling 80 would give us too many. So we noticed that if we added 20 JRs:52 JBs and 80 JRs:208 JBs, we would get 100 JRs:260 JBs.</td>
</tr>
<tr><th colspan="3">Group 8 (scaling up)</th></tr>
<tr><td colspan="3">We drew 100 JRs in groups of 5. Then we put 13 JBs with each group of 5 JRs. We then counted the number of JBs and found we had used 260 of them.</td></tr>
</table>

Effective Mathematics Teaching Practices
Establish mathematics goals to focus learning. *Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.*
Implement tasks that promote reasoning and problem solving. *Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.*
Use and connect mathematical representations. *Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.*
Facilitate meaningful mathematical discourse. *Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.*
Pose purposeful questions. *Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.*
Build procedural fluency from conceptual understanding. *Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.*
Support productive struggle in learning mathematics. *Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.*
Elicit and use evidence of student thinking. *Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.*



National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: Author.

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